A COMPLETE DETERMINATION OF RABINOWITSCH POLYNOMIALS

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Abstract. Let m be a positive integer and $f_m(x)$ be a polynomial of the form $f_m(x) = x^2 + x - m$. We call a polynomial $f_m(x)$ a Rabinowitsch polynomial if for $s = [\sqrt{m}]$ and consecutive integers $x = x_0, x_0 + 1, \cdots, x_0 + s - 1$, $|f_m(x)|$ is either 1 or prime. In this paper, we show that there are exactly 14 Rabinowitsch polynomials $f_m(x)$.

1. Introduction

Euler discovered that the polynomial $x^2 + x + 41$ is prime for any integer such that $0 \leq x \leq 39$ and remarked that there are few such polynomials. In [14], Rabinowitsch proved the following theorem.

Theorem (Rabinowitsch) *Let m be a positive integer. The polynomial $x^2 + x + m$ is prime for any integer x such that $0 \leq x \leq m - 2$ if and only if $4m - 1$ is square-free and the class number of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{1-4m})$ is equal to 1.*

The complete determination of such polynomials is done by the complete determination of imaginary quadratic fields with class number one [1] [17]. So we know that there are exactly 7 such polynomials, corresponding to $m = 1, 2, 3, 5, 11, 17$ and 41.

Later, many authors studied analogue of this for real quadratic fields. For examples, see [6] [7] [9] [12] [13]. Let m be a positive integer and $f_m(x)$ be a polynomial of the form $f_m(x) = x^2 + x - m$. We call a polynomial $f_m(x)$ a *Rabinowitsch polynomial* if for $s = [\sqrt{m}]$ and consecutive integers $x = x_0, x_0 + 1, \cdots, x_0 + s - 1$, $|f_m(x)|$ is either 1 or prime. In [6] [7], Byeon and Stark proved the following theorem.

Theorem (Byeon and Stark) *Let m be a positive integer. If the polynomial $f_m(x) = x^2 + x - m$ is a Rabinowitsch polynomial, then $4m + 1$ is square-free except $m = 2$ and the class number of the real quadratic*

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field $K = \mathbb{Q}(\sqrt{4m+1})$ is equal to 1. Moreover, every Rabinowitsch polynomial $f_m(x)$ is one of the following types.

(i) $x^2 + x - 2$,
(ii) $x^2 + x - t^2$, *where t is 1 or a prime,*
(iii) $x^2 + x - (t^2 + t + n)$, *where $-t < n \leq t$, $|n|$ is 1 or $|n| = \frac{2t+1}{3}$ is an odd prime.*

We note that the real quadratic fields

$$K = \mathbb{Q}(\sqrt{4t^2+1}),$$

corresponding to the type (ii) and

$$K = \mathbb{Q}(\sqrt{4(t^2+t+n)+1}) = \mathbb{Q}(\sqrt{(2t+1)^2+4n}),$$

corresponding to the type (iii) are so-called *Richaud-Degert type*. In [7], using determination of real quadratic fields of Richaud-Degert type with class number one, they showed that there are all 14 Rabinowitsch polynomials $f_m(x)$ with at most one possible exception and there is no exception under the the generalized Riemann hypothesis. The aim of this paper is to show the following theorem unconditionally.

Theorem 1.1. *There are exactly 14 Rabinowitsch polynomials $f_m(x)$. And the complete list of such (m, x_0) is $(m, x_0) = (1, 0)$, $(2, 0)$, $(3, 0)$, $(4, 1)$, $(7, 0)$, $(9, 1)$, $(13, 0)$, $(25, 1)$, $(43, 0)$, $(49, 1)$, $(73, 0)$, $(103, 4)$, $(169, 1)$, $(283, 6)$.*

In [2] [3], Biro completely determined the real quadratic fields $K = \mathbb{Q}(\sqrt{4t^2+1})$ and $K = \mathbb{Q}(\sqrt{(2t+1)^2+4})$ of class number one. In [5], Byeon, Kim and Lee completely determined the real quadratic fields $K = \mathbb{Q}(\sqrt{(2t+1)^2-4})$ of class number one. Using these results, we can completely determine Rabinowitsch polynomials of type (ii) and (iii) $|n| = 1$. Since the real quadratic fields corresponding to (iii) $|n| = \frac{2t+1}{3}$ is $K = \mathbb{Q}(\sqrt{9n^2 \pm 4n})$, if we prove the following theorem and completely determine the real quadratic fields $K = \mathbb{Q}(\sqrt{9n^2 \pm 4n})$ of class number one, we can obtain Theorem 1.1 from the table in [7].

Theorem 1.2. *Let $n \neq 1$ be a positive integer. Let $d = 9n^2 \pm 4n$ be a positive square-free integer and $h(d)$ the class number of the real quadratic field $K = \mathbb{Q}(\sqrt{d})$. Then $h(d) \geq 2$ if $n \geq 162871$.*

To porove Theorem 1.2, we basically follow Biro's method in [2] [3] and by using an observation in section 2, we simplify the computation of special values of L-functions.

2. Preliminaries

Let K be a real quadratic field and $\mathbf{f}$ an ideal of K. Let $I_K(\mathbf{f})$ be a group of ideal $\mathbf{a}$ of K with $(\mathbf{a},\mathbf{f}) = 1$ and $P_{\mathbf{f},1}$ be a subgroup of $I_K(\mathbf{f})$ consisting of principal ideals (α) generated by a totally positive $\alpha \in 1+\mathbf{f}$. Let $CL_K(\mathbf{f}) = I_K(\mathbf{f})/P_{\mathbf{f},1}$ be the ray class group of modulo $\mathbf{f}$ and χ be a ray class character. Let $N(\mathbf{a})$ be the number of $O(K)/\mathbf{a}$ for integral ideal $\mathbf{a}$ and $N_K(\alpha) = \alpha\bar{\alpha}$. Then the L-function of K and χ $L_K(s,\chi)$ is defined as follows;

$$L_K(s,\chi) := \sum_{\text{integral } \mathbf{a} \in I_K(\mathbf{f})} \chi(\mathbf{a}) N(\mathbf{a})^{-s}.$$

For a ray class $\mathbf{C}$, we define the partial zeta function $\zeta(s,\mathbf{C})$ as follows;

$$\zeta(s,\mathbf{C}) := \sum_{\text{integral } \mathbf{a} \in \mathbf{C}} N(\mathbf{a})^{-s}.$$

Let $\mathbf{C}_2$ be the ray class defined by

$$\mathbf{C}_2 = [(\mu_2)],\ \mu_2 \in 1+\mathbf{f},\ \mu_2 > 0, \mu_2' < 0.$$

Then we have the following proposition.

Proposition 2.1. *For a ray class* $\mathbf{C} \in CL_K(\mathbf{f})$,

$$\zeta(0,\mathbf{C}) + \zeta(0,\mathbf{C}_2\mathbf{C}) = 0.$$

Proof: Let $\mathbf{B}$ be a ray class and H be a group of ray class characters χ such that

$$\chi(\mathbf{C}_2) = 1.$$

We note that for $\chi \in H$,

$$L_K(s,\chi) = \sum_{\mathbf{C} \in CL_K(\mathbf{f})} \chi(\mathbf{C})\zeta(s,\mathbf{C}) = \sum_{\mathbf{C} \in CL_K(\mathbf{f})} \chi(\mathbf{C})\zeta(s,\mathbf{C}\mathbf{C}_2).$$

Thus we have

$$\begin{aligned}
& 2\sum_{\chi \in H} \chi(\mathbf{B}^{-1}) L_K(s,\chi) \\
= & \sum_{\chi \in H} \chi(\mathbf{B}^{-1}) \sum_{\mathbf{C} \in CL_K(\mathbf{f})} \chi(\mathbf{C}) \Big[\zeta(s,\mathbf{C}) + \zeta(s,\mathbf{C}_2\mathbf{C})\Big] \\
= & \sum_{\mathbf{C} \in CL_K(\mathbf{f})} \sum_{\chi \in H} \chi(\mathbf{B}^{-1}\mathbf{C}) \Big[\zeta(s,\mathbf{C}) + \zeta(s,\mathbf{C}_2\mathbf{C})\Big] \\
= & |H|\Big(\zeta(s,\mathbf{B}) + \zeta(s,\mathbf{C}_2\mathbf{B})\Big).
\end{aligned}$$

Since for $\chi \in H$,

$$L_K(0,\chi) = 0,$$

we complete the proof. □

Let $\{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_h\}$ be the complete representatives of narrow ideal class group of K. We define

$$L_K(\mathbf{b}, s, \chi) := \sum_{\text{integral } \mathbf{a} \sim \mathbf{b}} \chi(\mathbf{a}) N(\mathbf{a})^{-s},$$

where $\mathbf{a} \sim \mathbf{b}$ means that

$$\mathbf{a} = (\alpha)\mathbf{b},$$

for totally positive element α. Then we have

$$L_K(s,\chi) = \sum_{i=1}^{h} L_K(b_i, s, \chi).$$

Proposition 2.2. *If* $\chi(\mathbf{C}_2) = -1$ *then*

$$L_K(\mathbf{b}, 0, \chi) = L_K(\mu_2 \mathbf{b}, 0, \chi).$$

Proof: Let $\{\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_m\}$ be the complete representatives of ray classes modulo $\mathbf{f}$ such that $\mathbf{a}_i \sim \mathbf{b}$. Then from Proposition 2.1

$$\begin{aligned} L_K(\mu_2 \mathbf{b}, 0, \chi) &= \sum_{i=1}^{m} \chi(\mathbf{C}_2[\mathbf{a}_i])\zeta(0, \mathbf{C}_2[\mathbf{a}_i]) \\ &= \sum_{i=1}^{m} \chi([\mathbf{a}_i])\zeta(0, [\mathbf{a}_i]) = L_K(\mathbf{b}, 0, \chi). \end{aligned}$$

□

3. Computation of $L_K(0,\chi)$

3.1. **$K = \mathbb{Q}(\sqrt{9n^2+4n})$.** Let $d = 9n^2 + 4n$ be a positive square free integer. Let K be the real quadratic field $K = \mathbb{Q}(\sqrt{d})$ and $h(d)$ its class number. Then the fundamental unit of K is

$$\epsilon = \frac{9n + 2 + 3\sqrt{9n^2+4n}}{2}$$

and $\{1, \omega\}$ is a basis of a ring of integer O_K, where

$$\omega = \frac{3n + \sqrt{9n^2+4n}}{2}.$$

We note that

$$\omega = \frac{\epsilon - 1}{3}.$$

For an ideal $\mathbf{b}$ of K, we define

$R(\mathbf{b}) := \{x+y\epsilon | x, y \in \mathbb{Q} \text{ with } 0 < x \leq 1, 0 \leq y < 1 \text{ and } \mathbf{b}(x+y\epsilon) \subset O(K)\}$.

Then we find that

(1)

$$L_K(\mathbf{b}, 0, \chi) = \sum_{x+y\epsilon \,\in\, R(\mathbf{b})} \chi((x+y\epsilon)\mathbf{b}) \sum_{n_1,n_2=0}^{\infty} N_K(x+n_1+(y+n_2)\epsilon)^{-s}|_{s=0}.$$

To compute $L_K(\mathbf{b}, 0, \chi)$, we need the following lemmas.

Lemma 3.1 (Shintani [15] [16])**.**

$$\sum_{n_1,n_2=0}^{\infty} N_K(x+n_1+(y+n_2)\epsilon)^{-s}|_{s=0} = (x-\frac{1}{2})(y-\frac{1}{2})+\frac{1}{4}(\epsilon+\bar{\epsilon})(x^2+y^2-x-y+\frac{1}{3}).$$

Lemma 3.2. *Let q be a positive rational integer and $K = \mathbb{Q}(\sqrt{d})$, where $d = 9n^2 + 4n$ is a positive square free integer. Then*

$$\{(x, y) | x + y\epsilon \in R((q))\} = \{(x, y) | x = \delta_1(j) + \frac{C}{q} - \frac{D + qj}{3q} \text{ and } y = \frac{D + qj}{3q}\},$$

where $0 \leq j \leq 2$, $0 \leq C, D \leq q - 1$ and

$$\delta_1(j) = \begin{cases} 0, & 0 \leq j \leq B_{C,D} - 1 \\ 1, & B_{C,D} \leq j \leq 2 \end{cases}$$

for $B_{C,D} := -[\frac{D-3C}{q}]$.

Proof: Suppose that $x + y\epsilon \in R((q))$ and

$$q(x + y\epsilon) = C + D\omega + q(i + j\omega) = C + qi + (D + qj)\Big(\frac{\epsilon - 1}{3}\Big),$$

for $0 \leq C, D \leq q - 1$ and some integers i, j. Since

$$0 \leq y = \frac{D + qj}{3q} < 1,$$

we have

$$y = \frac{D + qj}{3q} \quad \text{for } j = 0, 1, 2.$$

And since

$$0 < x = \frac{C}{q} + i - \frac{D + qj}{3q} \leq 1,$$

we have

$$x = 1 + \left[\frac{D+qj}{3q} - \frac{C}{q}\right] + \frac{C}{q} - \frac{D+qj}{3q}$$
$$= \begin{cases} \frac{C}{q} - \frac{D+qj}{3q} & \text{for } 0 \leq j < -\left[\frac{D-3C}{q}\right] \\ 1 + \frac{C}{q} - \frac{D+qj}{3q} & \text{for } -\left[\frac{D-3C}{q}\right] \leq j \leq 2. \end{cases}$$

□

Let q be a positive rational integer and $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ be a odd primitive character with conductor q. Then we define

$$\chi \circ N : I_K(q) \to \mathbb{C}^*,$$

by $\chi \circ N(\mathbf{a}) = \chi(N(\mathbf{a}))$. Then $\chi \circ N$ is a ray class character modulo q since for $\mathbf{a} \in P_{q,1}$, we have

$$N(\mathbf{a}) \equiv 1 \pmod{q}.$$

Theorem 3.3. *Let q be a positive rational integer and $K = \mathbb{Q}(\sqrt{d})$, where $d = 9n^2 + 4n$ is a positive square free integer. Let $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ be a odd primitive character with conductor q. If $h(d) = 1$, then*

$$\begin{aligned} L_K(0, \chi\circ N) &= 2L_K((q), 0, \chi \circ N) \\ &= \frac{1}{2q^2} \sum_{0\leq C,D\leq q-1} \chi(C^2 + 3nCD - nD^2) \\ &\cdot (6C^2 + 27C^2n - 18CDn + 6D^2n \\ &- 4B_{C,D}Cq + 9Cnq - 18B_{C,D}Cnq - 6Dnq + 6B_{C,D}Dnq \\ &- q^2 + 2B_{C,D}q^2 + nq^2 - 3B_{C,D}nq^2 + 3B_{C,D}^2 nq^2) \end{aligned}$$

where $B_{C,D} = -[\frac{D-3C}{q}]$.

Proof: If $h(d) = 1$ then $\{(q), (q\mu_2)\}$ is a complete representatives of narrow ideal class group of K, since the fundamental unit ϵ has norm 1. And

$$\chi \circ N([\mu_2]) = \chi(-N_K(\mu_2)) = -1.$$

Thus from Proposition 2.2 we have

$$L_K(0, \chi \circ N) = 2L_K((q), 0, \chi \circ N).$$

If $x = \delta_1(j) + \frac{C}{q} - \frac{D+qj}{3q}$ and $y = \frac{D+qj}{3q}$ then

$$q(x + y\epsilon) \equiv C + D\omega \pmod{q}.$$

Thus

$$N(q(x + y\epsilon)) = N_K(q(x + y\epsilon)) \equiv C^2 + 3nCD - nD^2 \pmod{q}.$$

Now from the equation (1), Lemma 3.1 and Lemma 3.2, we can prove the theorem. □

Corollary 3.4. *Let q be a positive rational integer and $K = \mathbb{Q}(\sqrt{d})$, where $d = 9n^2 + 4n$ is a positive square free integer. Let $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ be a odd primitive character with conductor q. If $h(d) = 1$ and $n = qk + r$ for $0 \le r < q$, then*

$$L_K(0, \chi \circ N) = \frac{1}{2q^2}(B_\chi(r)k + A_\chi(r)),$$

where

$$\begin{aligned} A_\chi(r) &= \sum_{0 \le C,D \le q-1} \chi(C^2 + 3rCD - rD^2) \\ &\cdot \ (6C^2 + 27C^2r - 18CDr + 6D^2r \\ &- \ 4B_{C,D}Cq + 9Crq - 18B_{C,D}Crq - 6Drq + 6B_{C,D}Drq \\ &- \ q^2 + 2B_{C,D}q^2 + rq^2 - 3B_{C,D}rq^2 + 3B_{C,D}^2rq^2) \end{aligned}$$

$$\begin{aligned} B_\chi(r) &= \sum_{0 \le C,D \le q-1} \chi(C^2 + 3rCD - rD^2) \\ &\cdot \ (27C^2q - 18CDq + 6D^2q \\ &+ \ 9Cq^2 - 18B_{C,D}Cq^2 - 6Dq^2 + 6B_{C,D}Dq^2 \\ &+ \ q^3 - 3B_{C,D}q^3 + 3B_{C,D}^2q^3). \end{aligned}$$

Proof: Since χ has a conductor q, we have for $n = qk + r$

$$\chi(C^2 + 3nCD - nD^2) = \chi(C^2 + 3rCD - rD^2).$$

Thus Corollary 3.4 follows from Theorem 3.3. □

3.2. $K = \mathbb{Q}(\sqrt{9n^2 - 4n})$**.** Let $d = 9n^2 - 4n$ be a positive square free integer. Let K be the real quadratic field $K = \mathbb{Q}(\sqrt{d})$ and $h(d)$ its class number. Then the fundamental unit of K is

$$\epsilon = \frac{9n - 2 + 3\sqrt{9n^2 - 4n}}{2}$$

and $\{1, \omega\}$ is a basis of a ring of integer O_K, where

$$\omega = \frac{3n + \sqrt{9n^2 - 4n}}{2}.$$

We note that

$$\omega = \frac{\epsilon + 1}{3}.$$

Lemma 3.5. *Let q be a positive rational integer and $K = \mathbb{Q}(\sqrt{d})$, where $d = 9n^2 - 4n$ is a positive square free integer. Then*

$$\{(x,y)|x + y\epsilon \in R((q))\} = \{(x,y)|x = \delta_2(j) - \frac{s_{C,D}}{3q} + \frac{j}{3} \text{ and } y = \frac{D+qj}{3q}\},$$

where $0 \le j \le 2$, $0 \le C, D \le q-1$ and

$$\delta_2(j) = \begin{cases} 1, & 0 \le j \le A_{C,D} \\ 0, & A_{C,D}+1 \le j \le 2 \end{cases}$$

for $s_{C,D} := -3C - D - 3q[\frac{-3C-D}{3q}]$ and $A_{C,D} := [\frac{s_{C,D}}{q}]$.

Proof: Suppose that $x + y\epsilon \in R((q))$ and

$$q(x+y\epsilon) = C + D\omega + q(i + j\omega) = C + qi + (D+qj)\Big(\frac{\epsilon+1}{3}\Big),$$

for $0 \le C, D \le q-1$ and some integers i, j. Since

$$0 \le y = \frac{D+qj}{3q} < 1,$$

we have

$$y = \frac{D+qj}{3q} \quad \text{for } j = 0, 1, 2.$$

And since

$$0 < x = \frac{C}{q} + i - \frac{D+qj}{3q} \le 1$$

we have for $s_{C,D} = -3C - D - 3q[\frac{-3C-D}{3q}]$

$$\begin{aligned} x &= 1 + \Big[-\frac{D+qj}{3q} - \frac{C}{q}\Big] + \frac{C}{q} + \frac{D+qj}{3q} \\ &= 1 + \Big[\frac{s_{C,D}}{3q} + [\frac{-3C-D}{3q}] - \frac{j}{3}\Big] - (\frac{s_{C,D}}{3q} + [\frac{-3C-D}{3q}] - \frac{j}{3}) \\ &= 1 + \Big[\frac{s_{C,D}}{3q} - \frac{j}{3}\Big] - (\frac{s_{C,D}}{3q} - \frac{j}{3}) \\ &= \begin{cases} 1 - \frac{s_{C,D}}{3q} - \frac{j}{3} & \text{for } 0 \le j \le \Big[\frac{s_{C,D}}{q}\Big] \\ -\frac{s_{C,D}}{3q} + \frac{j}{3} & \text{for } \Big[\frac{s_{C,D}}{q}\Big] + 1 \le j \le 2. \end{cases} \end{aligned}$$

□

Theorem 3.6. *Let q be a positive rational integer and $K = \mathbb{Q}(\sqrt{d})$, where $d = 9n^2 - 4n$ is a positive square free integer. Let $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ be a odd primitive character with conductor q. If $h(d) = 1$, then*

$$\begin{aligned} L_K(0, \chi \circ N) &= 2L_K((q), 0, \chi \circ N) \\ &= \frac{1}{6q^2} \sum_{0 \le C,D \le q-1} \chi(C^2 + 3nCD + nD^2) \\ &\cdot (-2D^2 + 9D^2 n + 4Dq + 4A_{C,D}Dq - 9Dnq \\ &- 3q^2 - 6A_{C,D}q^2 + 3nq^2 + 9A_{C,D}nq^2 + 9A_{C,D}^2 nq^2 \\ &- 4Ds_{C,D} + 4qs_{C,D} + 4A_{C,D}qs_{C,D} - 9nqs_{C,D} \\ &- 18A_{C,D}nqs_{C,D} - 2s_{C,D}^2 + 9ns_{C,D}^2) \end{aligned}$$

where $s_{C,D} = -3C - D - 3q[\frac{-3C-D}{3q}]$ and $A_{C,D} = [\frac{s_{C,D}}{q}]$.

Proof: If $h(d) = 1$ then $\{(q), (q\mu_2)\}$ is a complete representatives of narrow ideal class group of K, since the fundamental unit ϵ has norm 1. And

$$\chi \circ N([\mu_2]) = \chi(-N_K(\mu_2)) = -1.$$

Thus form Proposition 2.2 we have

$$L_K(0, \chi \circ N) = 2L_K((q), 0, \chi \circ N).$$

If $x = \delta_2(j) - \frac{s_{C,D}}{3q} + \frac{j}{3}$ and $y = \frac{D+qj}{3q}$ then

$$q(x + y\epsilon) \equiv C + D\omega \pmod{q}.$$

Thus

$$N(q(x + y\epsilon)) = N_K(q(x + y\epsilon)) \equiv C^2 + 3nCD + nD^2 \pmod{q}.$$

Now from the equation (1), Lemma 3.1 and Lemma 3.5, we can prove the theorem. □

Corollary 3.7. *Let q be a positive rational integer and $K = \mathbb{Q}(\sqrt{d})$, where $d = 9n^2 - 4n$ is a positive square free integer. Let $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ be a odd primitive character with conductor q. If $h(d) = 1$ and $n = qk + r$ for $0 \le r < q$, then*

$$L_K(0, \chi \circ N) = \frac{1}{6q^2}(F_\chi(r)k + E_\chi(r)),$$

where

$$\begin{aligned}
E_\chi(r) &= \sum_{0\le C,D\le q-1} \chi(C^2+3rCD+rD^2)\\
&\cdot\ (-2D^2+9D^2r+4Dq+4A_{C,D}Dq-9Drq\\
&-\ 3q^2-6A_{C,D}q^2+3rq^2+9A_{C,D}rq^2+9A_{C,D}^2rq^2\\
&-\ 4Ds_{C,D}+4qs_{C,D}+4A_{C,D}qs_{C,D}-9rqs_{C,D}\\
&-\ 18A_{C,D}rqs_{C,D}-2s_{C,D}^2+9rs_{C,D}^2)
\end{aligned}$$

$$\begin{aligned}
F_\chi(r) &= \sum_{0\le C,D\le q-1} \chi(C^2+3rCD+rD^2)\\
&\cdot\ (9D^2q-9Dq^2+3q^3+9A_{C,D}q^3+9A_{C,D}^2q^3\\
&-\ 9q^2s_{C,D}-18A_{C,D}q^2s_{C,D}+9qs_{C,D}^2).
\end{aligned}$$

Proof: Since χ has a conductor q, we have for $n = qk + r$

$$\chi(C^2+3nCD+nD^2) = \chi(C^2+3rCD+rD^2).$$

Thus Corollary 3.7 follows from Theorem 3.6. □

4. $q \to p$

Let χ be an odd primitive character with conductor q with $(q, d) = 1$ and L_χ a field over $\mathbb{Q}$ generated by the values of $\chi(a)$ for $a = 1, 2, \cdots q$ and $m_\chi := \sum_{a=1}^{q} a\chi(a)$.

Condition(*): *The integer q is odd, p is an odd prime, and there is an odd prime character χ with conductor q and a prime ideal I of L_χ lying over p such that $m_\chi \in I$ and the residue field of I is a prime field.*

We will denote by $p \to q$ that q, p satisfy Condition(*). From Section 4 in [2], we have

$$175 \to 61,\ 61 \to 1861,\ 175 \to 1861.$$

And from Section 4 in [10], we have

$$175 \to 601.$$

To prove Theorem 1.2, we need the following another $p \to q$.

Lemma 4.1. $175 \to 271$.

Proof: Consider the function $f_{25} : (\mathbb{Z}/25\mathbb{Z})^* \to \mathbb{Z}/20\mathbb{Z}$ for which $2^{f_{25}(a)} \equiv a \pmod{25}$ and the function $g_7 : (\mathbb{Z}/7\mathbb{Z})^* \to \mathbb{Z}/6\mathbb{Z}$ for which $3^{g_7(a)} \equiv a \pmod 7$. Above two functions are well defined, since $(\mathbb{Z}/25\mathbb{Z})^*$ [resp. $(\mathbb{Z}/7\mathbb{Z})^*$] is a cyclic group generated by 2 [resp. 3]. Define $\chi_5 : (\mathbb{Z}/175\mathbb{Z})^* \to \mathbb{C}$ by

$$\chi_5(a) = \zeta_{30}^{21 f_{25}(a_{25})} \cdot \zeta_{30}^{25 g_7(a_7)},$$

where $a \equiv a_{25} \pmod{25}$, $a \equiv a_7 \pmod 7$ and ζ_{30} is a primitive 30-th root of unity. Then χ_5 is an odd primitive character with a conductor 175. Since the order of 214 modulo 271 is 30, $I_5 = (271, \zeta_{30} - 214)$ is the prime ideal in $L_{\chi_5} = \mathbb{Q}(\zeta_{30})$ lying over rational prime 271 of degree 1 (See page 97 in [2]). From

$$\zeta_{30} \equiv 214 \pmod{I_5},$$

we find that

$$m_{\chi_5} \equiv 0 \pmod{I_5}.$$

So we obtain

$$175 \to 271.$$

□

5. Residues of n

5.1. $K = \mathbb{Q}(\sqrt{9n^2+4n})$. Let $d = 9n^2 + 4n$ be a positive square free integer. Let K be the real quadratic field $K = \mathbb{Q}(\sqrt{d})$ and $h(d)$ its class number. We assume that $h(d) = 1$. If integers q and p satisfy the Condition(*), then for r such that $B_\chi(r) \notin I$, there exists a unique $T_\chi(r) \in \{0, 1, 2, \cdots p-1\}$ such that

$$-q\frac{A_\chi(r)}{B_\chi(r)} + r + I = T_\chi(r) + I.$$

Thus we have

$$n \equiv T_\chi(r) \pmod p \quad \text{for } n = qk + r.$$

We define the functions $T_{\chi_i}(r)$ as follows:

$$-q_i\frac{A_{\chi_i}(r)}{B_{\chi_i(r)}} + r + I_i = T_{\chi_i}(r) + I_i,$$

where the characters χ_i and ideals I_i are defined in Example 1, Example 3 and Example 2 of Section 4 in [2], respectively for $i = 1, 2, 3$, χ_4 and I_4 are in section 4 in [10], χ_5 and I_5 are in Lemma 4.1 and q_i is the conductor of χ_i.

For a residue a_{175} modulo 175 with $B_{\chi_1}(a_{175}) \notin I_1$[resp. $B_{\chi_3}(a_{175}) \notin I_3$], we define b_{61}[resp. d_{1861}] by residues modulo 61[resp. 1861] for which

$$b_{61} = T_{\chi_1}(a_{175})$$

$$d_{1861} = T_{\chi_3}(a_{175}).$$

And for a residue b_{61} modulo 61 with $B_{\chi_2}(b_{61}) \notin I_2$, we define c_{1861} by a residue modulo 1861 such that

$$c_{1861} = T_{\chi_2}(b_{61}).$$

We define $N_{175}(9n^2 + 4n) := \{n \in \mathbb{Z}^+ \mid (9n^2 + 4n, 175) = 1\}$. By computer work, we find that for $a_{175} \in N_{175}(9n^2 + 4n)$ we have

$$B_{\chi_1}(a_{175}) \notin I_1,\ B_{\chi_3}(a_{175}) \notin I_3$$

and for $a_{175} \in N_{175}(9n^2 + 4n)$ with $a_{175} \neq 16, 132$, we have

$$B_{\chi_2}(T_{\chi_1}(a_{175})) \notin I_2.$$

Thus we have the following table for $a_{175} \in N_{175}(9n^2 + 4n)$.

a_{175}	b_{61}	c_{1861}	d_{1861}	a_{175}	b_{61}	c_{1861}	d_{1861}	a_{175}	b_{61}	c_{1861}	d_{1861}
1	1	1	1	2	2	2	2	3	3	3	3
6	3	3	1608	8	51	31	1807	11	11	11	11
13	18	1179	485	16	47		1572	17	14	1210	382
18	49	1062	1646	22	17	1842	1669	23	43	386	617
27	12	175	383	31	22	258	858	32	41	1241	1056
36	35	1733	1357	37	60	1860	406	38	5	5	1712
41	12	175	542	43	16	456	96	46	48	1317	334
48	59	1859	1159	51	13	566	810	52	38	1355	1025
53	58	1858	1216	57	36	1010	660	58	20	620	1476
62	49	1062	154	66	16	456	161	67	34	1187	1628
71	4	947	604	72	31	5	1119	73	38	1355	1309
76	7	222	1108	78	49	1062	1829	81	37	1297	950
83	24	1106	160	86	20	620	227	87	7	222	1607
88	30	1196	51	92	6	510	1195	93	22	258	1008
97	34	1187	575	101	23	1392	329	102	54	1854	875
106	26	1302	1577	107	29	1685	532	108	8	1036	49
111	34	1187	406	113	37	1297	49	116	40	1240	1084
118	32	14	1314	121	24	1106	838	122	5	5	79
123	7	222	102	127	26	1302	1090	128	7	222	730
132	47		190	136	6	510	171	137	35	1733	910
141	8	1036	2	142	24	1106	142	143	1	1	710
146	20	620	1386	148	58	1858	1208	151	44	911	1199
153	15	1400	392	156	23	1392	1333	157	2	2	1781
158	32	14	65	162	30	1196	1637	163	52	1044	1091
167	39	424	362	171	57	1857	1547	172	58	1858	171

5.2. $K = \mathbb{Q}(\sqrt{9n^2 - 4n})$**.** Let $d = 9n^2 - 4n$ be a positive square free integer. Let K be the real quadratic field $K = \mathbb{Q}(\sqrt{d})$ and $h(d)$ its class number. We assume that $h(d) = 1$. If integers q and p satisfy the Condition(*), then for r such that $F_\chi(r) \notin I$, there exists a unique $S_\chi(r) \in \{0, 1, 2, \cdots p-1\}$ such that

$$-q\frac{E_\chi(r)}{F_\chi(r)} + r + I = S_\chi(r) + I.$$

Thus we have

$$n \equiv S_\chi(r) \pmod{p} \quad \text{for } n = qk + r.$$

And we define the functions $S_{\chi_i}(r)$ as follows:

$$-q_i\frac{E_{\chi_i}(r)}{F_{\chi_i(r)}} + r + I_i = S_{\chi_i}(r) + I_i,$$

where the characters χ_i and ideals I_i are defined in Example 1, Example 3 and Example 2 of Section 4 in [2], respectively for $i = 1, 2, 3$, χ_4 and I_4 are in section 4 in [10], χ_5 and I_5 are in Lemma 4.1 and q_i is the conductor of χ_i.

For a residue a_{175} modulo 175 with $F_{\chi_1}(a_{175}) \notin I_1$[resp. $F_{\chi_3}(a_{175}) \notin I_3$], we define b_{61}[resp. d_{1861}] by residues modulo 61[resp. 1861] for which

$$b_{61} = S_{\chi_1}(a_{175})$$

$$d_{1861} = S_{\chi_3}(a_{175}).$$

And for a residue b_{61} modulo 61 with $F_{\chi_2}(b_{61}) \notin I_2$, we define c_{1861} by a residue modulo 1861 such that

$$c_{1861} = S_{\chi_2}(b_{61}).$$

We define $N_{175}(9n^2 - 4n) := \{n \in \mathbb{Z}^+ \mid (9n^2 - 4n, 175) = 1\}$. By computer work, we find that for $a_{175} \in N_{175}(9n^2 - 4n)$ we have

$$F_{\chi_1}(a_{175}) \notin I_1, \; F_{\chi_3}(a_{175}) \notin I_3$$

and for $a_{175} \in N_{175}(9n^2 - 4n)$ with $a_{175} \neq 43, 159$, we have

$$F_{\chi_2}(S_{\chi_1}(a_{175})) \notin I_2.$$

Thus we have the following table for $a_{175} \in N_{175}(9n^2 - 4n)$.

a_{175}	b_{61}	c_{1861}	d_{1861}	a_{175}	b_{61}	c_{1861}	d_{1861}	a_{175}	b_{61}	c_{1861}	d_{1861}
3	3	3	3	4	4	4	4	8	22	1437	1819
12	9	817	663	13	31	665	1547	17	29	1847	1534
18	59	1859	140	19	38	469	1371	22	46	461	1231
24	17	950	724	27	3	3	1023	29	41	1241	363
32	60	1860	1169	33	37	755	1779	34	53	825	1303
38	26	128	1841	39	55	1351	1737	43	14		616
47	54	1639	442	48	35	559	1529	52	54	1639	834
53	56	1856	1842	54	37	755	713	57	29	1847	1680
59	21	621	1134	62	24	564	1705	64	27	674	1451
67	53	825	1004	68	32	176	1389	69	35	559	293
73	7	7	15	74	38	469	1299	78	27	674	833
82	39	1603	684	83	55	1351	837	87	31	665	369
88	54	1639	314	89	41	1241	309	92	37	755	985
94	24	564	198	97	12	799	944	99	54	1639	25
102	23	506	242	103	30	1856	802	104	57	914	295
108	27	674	1222	109	45	1405	497	113	12	799	1197
117	41	1241	123	118	25	851	544	122	3	3	1183
123	23	506	896	124	48	1295	1030	127	2	2	1612
129	13	544	1270	132	45	1409	1767	134	49	1686	1081
137	56	1856	78	138	1	1	1515	139	26	128	1028
143	20	620	71	144	39	1603	1415	148	49	1686	602
152	18	1475	934	153	44	19	847	157	12	799	1509
158	47	651	1539	159	14		1338	162	43	682	120
164	50	1850	1387	167	10	1830	1586	169	58	1858	858
172	58	1858	1469	173	59	1859	58	174	60	1860	774

6. Proof of Theorem 1.2

6.1. $K = \mathbb{Q}(\sqrt{9n^2+4n})$**.** Let $d = 9n^2 + 4n$ be a positive square free integer. Let K be the real quadratic field $K = \mathbb{Q}(\sqrt{d})$ and $h(d)$ its class number. From [Corollary 3.20, 4] (in [Corollary 3.20, 4], $r|n-t$ should be corrected by $r \not| n-t$ and $r(\frac{m^2}{4} - k^2 - k - 1) + 1$ should be corrected by $r(\frac{m^2-k^2-k-1}{4}) + 1$), we have the following lemma.

Lemma 6.1. *Let $d = 9n^2 + 4n$ be a positive square free integer and $n = \frac{2t+1}{3}$. Then*

$$h(d) = 1 \;\Leftrightarrow\; \frac{9n^2}{4} + n - \frac{1}{4} - x^2 - x \;(0 \le x \le t,\; x \ne t-n) \quad \textit{and}$$
$$2n+1 \quad \textit{are primes.}$$

Proposition 6.2. *If* 5 *or* 7 *or* 61 *divide* $d = 9n^2+4n$ *for an odd integer* n *then* $h(d) > 1$.

Proof: If 5 divides $9n^2 + 4n$, then 5 divides n or $9n + 4$. If 5 divides n then $n = 5$, since n is a prime number. But the class number of $\mathbb{Q}(\sqrt{245})$ is not 1. Thus 5 must divide $(9n + 4)$. Since $5 \neq 9n + 4$ for any prime n, we have $9n + 4 = 5k$ for some integer $k > 1$. Thus $9n^2 + 4n$ has at least 3 prime factors and $h(d) > 1$.

In this way, we can prove that if 7 or 61 divide $9n^2 + 4n$ for an odd integer n then $h(d) > 1$. □

Proposition 6.3. *If* $n \not\equiv 1, 2, 3, 11 \pmod{175}$ *then* $h(d) > 1$.

Proof: For $n \notin N_{175}(9n^2 + 4n)$, we find that $h(d) > 1$, from Proposition 6.2.

Now we consider $n \in N_{175}(9n^2 + 4n)$. Let $n \equiv 16, 132 \pmod{175} \in N_{175}(9n^2 + 4n)$ and $h(9n^2 + 4n) = 1$. Then from the table in section 5.1, we find that

$$n \equiv 47 \pmod{61}.$$

If $n \equiv 47 \pmod{61}$, 61 divides $9n^2 + 4n$. By Proposition 6.2, it is impossible.

Let $n \not\equiv 1, 2, 3, 11, 16, 132 \pmod{175} \in N_{175}(9n^2 + 4n)$ and $h(9n^2 + 4n) = 1$. Then from the table in section 5.1, we find that

$$c_{1861} \neq d_{1861}.$$

It is a contradiction. And this completes the proof. □

Proposition 6.4. *If* n *is an odd integer with* $n \equiv 1 \pmod{175}$ *and* $h(d) = 1$ *then* $[\sqrt{\frac{9n^2}{4} + n - \frac{1}{4}}] \leq 36661$.

Proof: From the table in section 5.1, we have

$$n \equiv 1 \pmod{61}. \tag{2}$$

Since $T_{\chi_4}(1) = 1$, we also have

$$n \equiv 1 \pmod{601}. \tag{3}$$

If $x = 36661l + 28890$ then form (2), we find that $x^2 + x - (\frac{9n^2}{4} + n - \frac{1}{4})$ is a multiple of 61 and from (3), we also find that $x^2 + x - (\frac{9n^2}{4} + n - \frac{1}{4})$ is a multiple of 601. Thus $x^2 + x - (\frac{9n^2}{4} + n - \frac{1}{4})$ is a multiple of 36661 for $x = 36661l + 28890$. We note that $x = 36661l + 28890$ can not be equal to $t - n$ because $t - n \equiv 0 \pmod{61}$ from (2). Thus from Lemma 6.1, if $t = [\sqrt{\frac{9n^2}{4} + n - \frac{1}{4}}] > 36661$, then $h(d) > 1$. □

Proposition 6.5. *If n is an odd integer with $n \equiv 2 \pmod{175}$ and $h(d) = 1$, then* $[\sqrt{\frac{9n^2}{4} + n - \frac{1}{4}}] \leq 35$.

Proof: If $x = 35l + 1$, then $x^2 + x - (\frac{9n^2}{4} + n - \frac{1}{4})$ is a multiple of 35. We note that $x = 35l + 1$ can not be equal to $t - n$ because $t - n \equiv 3 \pmod 5$. Thus from Lemma 6.1, if $t = [\sqrt{\frac{9n^2}{4} + n - \frac{1}{4}}] > 35$, then $h(d) > 1$. □

Proposition 6.6. *If n is an odd integer with $n \equiv 3 \pmod{175}$ and $h(d) = 1$, then* $[\sqrt{\frac{9n^2}{4} + n - \frac{1}{4}}] \leq 13027$.

Proof: From the table in section 5.1, we have

$$n \equiv 3 \pmod{1861}. \tag{4}$$

If $x = 13027l + 8079$ then form $n \equiv 3 \pmod 7$, we find that $x^2 + x - (\frac{9n^2}{4} + n - \frac{1}{4})$ is a multiple of 7 and from (4), we also find that $x^2+x-(\frac{9n^2}{4}+n-\frac{1}{4})$ is a multiple of 1861. Thus $x^2+x-(\frac{9n^2}{4}+n-\frac{1}{4})$ is a multiple of 13027 for $x = 13027l+8079$. We note that $x = 13027l+8079$ can not be equal to $t-n$ because $t-n \equiv 1 \pmod{1861}$ from (4). Thus from Lemma 6.1, if $t = [\sqrt{\frac{9n^2}{4} + n - \frac{1}{4}}] > 13027$, then $h(d) > 1$. □

Proposition 6.7. *If n is an odd integer with $n \equiv 11 \pmod{175}$ and $h(d) = 1$, then* $[\sqrt{\frac{9n^2}{4} + n - \frac{1}{4}}] \leq 162871$.

Proof: Since $T_{\chi_4}(11) = 11$, we have

$$n \equiv 11 \pmod{601}. \tag{5}$$

And from $T_{\chi_5}(11) = 11$, we also have

$$n \equiv 11 \pmod{271}. \tag{6}$$

If $x = 162871l + 5152$ then form (5), we find that $x^2 + x - (\frac{9n^2}{4} + n - \frac{1}{4})$ is a multiple of 601 and from (6), we also find that $x^2 + x - (\frac{9n^2}{4} + n - \frac{1}{4})$ is a multiple of 271. Thus $x^2 + x - (\frac{9n^2}{4} + n - \frac{1}{4})$ is a multiple of 162871 for $x = 162871l + 5152$. We note that $x = 162871l + 5152$ can not be equal to $t - n$ because $t - n \equiv 5 \pmod{601}$ from (5). Thus from Lemma 6.1, if $t = [\sqrt{\frac{9n^2}{4} + n - \frac{1}{4}}] > 162871$, then $h(d) > 1$. □

By combining Proposition 6.3-6.7, we have the following theorem:

Theorem 6.8. *Let $d = 9n^2+4n$ be a positive square-free integer. Then $h(d) \geq 2$ if $n \geq 162871$.*

6.2. $K = \mathbb{Q}(\sqrt{9n^2-4n})$. Let $d = 9n^2 - 4n$ be a positive square free integer. Let K be the real quadratic field $K = \mathbb{Q}(\sqrt{d})$ and $h(d)$ its class number. From [Corollary 3.21, 4] (in [Corollary 3.21, 4], $r|n-t$ should be corrected by $r \not| n-t$ and $r(\frac{m^2}{4} - k^2 - k - 1) - 1$ should be corrected by $r(\frac{m^2-k^2-k-1}{4}) - 1$), we have the following lemma.

Lemma 6.9. *Let $d = 9n^2 + 4n$ be a positive square free integer and $n = \frac{2t+1}{3}$. Then*

$$h(d) = 1 \quad \Leftrightarrow \quad \frac{9n^2}{4} - n - \frac{1}{4} - x^2 - x \ (0 \le x \le t,\ x \ne t-n) \quad \text{and} \quad 2n-1 \quad \text{are primes.}$$

Proposition 6.10. *If 5 or 7 or 61 divide $9n^2 - 4n$ for an odd integer n, then $h(d) > 1$ except for $d = 413$.*

Proof: If 7 divides $9n^2 - 4n$, then 7 divides n or $9n - 4$. If 7 divides n then $n = 7$, since n is a prime number and $d = 413$. We note that $h(413) = 1$. If 7 divides $(9n - 4)$, then since $7 \ne 9n - 4$ for any prime n, we have $9n - 4 = 7k$ for some integer $k > 1$. Thus $9n^2 - 4n$ has at least 3 prime factors and $h(d) > 1$.

In this way, we can prove that if 5 or 61 divide $9n^2 + 4n$ for an odd integer n then $h(d) > 1$. □

Proposition 6.11. *If $n \not\equiv 3, 4 \pmod{175}$ then $h(d) > 1$.*

Proof: For $n \notin N_{175}(9n^2 - 4n)$, we find that $h(d) > 1$, from Proposition 6.10.

Now we consider $n \in N_{175}(9n^2 - 4n)$. Let $n \equiv 43, 159 \pmod{175} \in N_{175}(9n^2 - 4n)$ and $h(9n^2 - 4n) = 1$. Then from the table in section 5.2, we find that

$$n \equiv 14 \pmod{61}.$$

If $n \equiv 14 \pmod{61}$, 61 divides $9n^2 - 4n$. By Proposition 6.10, it is impossible.

Let $n \not\equiv 3, 4, 43, 159 \pmod{175} \in N_{175}(9n^2 - 4n)$ and $h(9n^2 - 4n) = 1$. Then from the table in section 5.2, we find that

$$c_{1861} \ne d_{1861}.$$

It is an contradiction. And this completes the proof. □

Proposition 6.12. *If n is an odd integer with $n \equiv 3 \pmod{175}$ and $h(d) = 1$, then $[\sqrt{\frac{9n^2}{4} - n - \frac{1}{4}}] \le 3005$.*

Proof: Since $S_{\chi_4}(3) = 3$, we also have

$$n \equiv 3 \pmod{601}. \tag{7}$$

If $x = 3005l+1411$ then form (7), we find that $x^2+x-(\frac{9n^2}{4}-n-\frac{1}{4})$ is a multiple of 601. Since $n \equiv 3 \pmod{10}$, we also find that $x^2+x-(\frac{9n^2}{4}-n-\frac{1}{4})$ is a multiple of 5 for $x = 3005l+1411$. Thus $x^2+x-(\frac{9n^2}{4}-n-\frac{1}{4})$ is a multiple of 3005 for $x = 3005l+1411$. We note that $x = 3005l+1411$ can not be equal to $t-n$ because $t-n \equiv 1 \pmod{601}$ from (7). Thus from Lemma 6.9, if $t = [\sqrt{\frac{9n^2}{4}+n-\frac{1}{4}}] > 3005$, then $h(d) > 1$. □

Proposition 6.13. *If n is an odd integer with $n \equiv 4 \pmod{175}$ and $h(d) = 1$, then $[\sqrt{\frac{9n^2}{4}-n-\frac{1}{4}}] \leq 4207$.*

Proof: Since $S_{\chi_4}(4) = 4$, we also have

$$n \equiv 4 \pmod{601}. \tag{8}$$

If $x = 4207l+3018$ then form (8), we find that $x^2+x-(\frac{9n^2}{4}-n-\frac{1}{4})$ is a multiple of 601. Since $n \equiv 11 \pmod{14}$, we also find that $x^2+x-(\frac{9n^2}{4}-n-\frac{1}{4})$ is a multiple of 7 for $x = 4207l+3018$. Thus $x^2+x-(\frac{9n^2}{4}-n-\frac{1}{4})$ is a multiple of 4207 for $x = 4207l+3018$. We note $x = 4207l+3018$ that can not be equal to $t-n$ because $t-n \equiv 302 \pmod{601}$ from (8). Thus from Lemma 6.9, if $t = [\sqrt{\frac{9n^2}{4}+n-\frac{1}{4}}] > 4207$, then $h(d) > 1$. □

By combining Proposition 6.11-6.13, we have the following theorem:

Theorem 6.14. *Let $d = 9n^2 - 4n$ be a positive square-free integer. Then $h(d) \geq 2$ if $n \geq 4207$.*

Proof of Theorem 1.2: Theorem 1.2 follows from Theorem 6.8 and 6.14. □

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